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The homology of certain subgroups of the symmetric group with coefficients in $\mathscr{L}ie(n)$

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Abstract

The authors compute the mod p homology of groups of the form $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ with coefficients in $\mathscr{L}ie(n)$, where $n_1 + \cdots + n_k = n$.

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0. Introduction

For a commutative ring R and a positive integer n, let $\mathscr{L}ie_R(x_1,\ldots,x_n)$ be the free Lie algebra over R generated by $x_1,\ldots,x_n$. Let $\mathscr{L}ie_R(n) \subset \mathscr{L}ie_R(x_1,\ldots,x_n)$ be the submodule spanned by all bracket monomials containing each x_i exactly once. In what follows, R will almost always be $\mathbb{Z}$ or $\mathbb{Z}/p$ for some prime p, and we often suppress R from the notation. Whenever we perform explicit calculations, we assume $R = \mathbb{Z}/p$, otherwise, $R = \mathbb{Z}$. The submodule $\mathscr{L}ie(n)$ is invariant under the obvious action of the symmetric group Σ_n on $\mathscr{L}ie(x_1,\ldots,x_n)$. We consider $\mathscr{L}ie(n)$ as a representation of Σ_n. It is well-known that as an R-module

$$\mathscr{L}ie_R(n) \cong \bigoplus_{(n-1)!} R.$$

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As a Σ_n-representation, $\mathscr{L}ie(n)$ cannot, in general, be described so trivially. It is known (see, for instance, [9]) that if $R = \mathbb{C}$ then

$$\mathscr{L}ie(n) \cong \hom(\mathrm{Ind}_{\mathbb{Z}/n}^{\Sigma_n} \Psi, \mathbb{C}[-1]),$$

where Ψ is the representation of $\mathbb{Z}/n$ sending the generator to $e^{2\pi i/n}$ and $\mathbb{C}[-1]$ is the sign representation. However, if the ground ring is $\mathbb{Z}$ or $\mathbb{Z}/p$ then it is easy to check directly that $\mathscr{L}ie(n)$ is not closely related to any representation induced from a one-dimensional representation of $\mathbb{Z}/n$. Following a suggestion of the referee, we also remark that if $p|n$ then $\mathscr{L}ie(n)$ cannot be projective in characteristic p since projectivity in characteristic p would imply freeness over the p-Sylow subgroup of Σ_n which cannot happen since the order of the p-Sylow subgroup does not divide $(n-1)!$ when p divides n.

Let $n_1, \ldots, n_k$ be positive integers such that $n = n_1 + \cdots + n_k$. Consider the group $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ as a subgroup of Σ_n. The homology groups $H_*(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}; \mathscr{L}ie(n))$ arise in various contexts in mathematics. In particular, they arise as basic building blocks for homology of spaces of unordered configurations of points in manifolds and for homology of Lie algebras (see, for instance, [2–4]). The homology groups $H_*(\Sigma_n; \mathscr{L}ie(n))$ (with the ground ring $\mathbb{Z}/p$) were computed in [1] (in a certain guise, as will be explained below). The main purpose of this paper is to use this computation to describe

$$H_*(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}; \mathscr{L}ie(n))$$

(with the same ground ring).

Thus, the main result of this paper is a recursive formula (Theorem 0.2) which reduces computation of $H_*(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}; \mathscr{L}ie(n))$ to computation of $H_*(\Sigma_d; \mathscr{L}ie(d))$ for various values of d. Our methods involve some homotopy theory, namely the Hilton–Milnor theorem and Goodwillie's calculus of functors ([5–7]). In fact, we do a little more than prove Theorem 0.2 in the sense that rather than prove that two graded vector spaces are isomorphic, we prove that two spectra are homotopy-equivalent and then obtain the result on vector spaces by passing to homology. We use that the nth derivative (in the sense of Goodwillie) of the functor ΩS (loop-suspension) is a spectrum with an action of Σ_n, whose homology is concentrated in one degree, and in this degree it is precisely $\mathscr{L}ie(n)$. In more detail, we consider the Goodwillie tower (the Taylor tower of [7]) of the multi-variable functor $\Omega S(X_1 \vee X_2 \vee \cdots \vee X_k)$ and we evaluate its differentials in two different ways. The first way is to consider the Goodwillie tower of the functor ΩS and evaluate it at $X_1 \vee X_2 \vee \cdots \vee X_k$. The nth layer is given, as a spectrum, by the following formula (we denote $W_{h\Sigma_n} = W \wedge_{\Sigma_n} E\Sigma_{n+}$ for any space or spectrum W):

$$\mathrm{Map}_*(S^2 K_n, \Sigma^\infty (SX_1 \vee SX_2 \vee \cdots \vee SX_k)^{\wedge n})_{h\Sigma_n},$$

where K_n is a complex with an action of Σ_n, that is non-equivariantly homotopy equivalent to a wedge of $(n-1)!$ copies of the $(n-2)$-dimensional sphere, and whose only non-trivial reduced homology group satisfies

$$\hom(H_{n-2}(K_n), \mathbb{Z}[-1]) \cong_{\mathbb{Z}[\Sigma_n]} \mathscr{L}ie(n).$$

This is proved as Fact 2.3 below. K_n is essentially the geometric realization of the poset of partitions of the set with n elements (the precise definition is given in Section 2). By using the binomial expansion, it should be intuitively clear (to anyone familiar with the Goodwillie calculus) that the $(n_1, \dots, n_k)$-differential of the functor $\Omega S(X_1 \vee X_2 \vee \cdots \vee X_k)$, which is the counterpart of

$$\frac{\partial^n f}{\partial x_1^{n_1} \cdots \partial x_k^{n_k}}(0, \dots, 0)\frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1! \cdots n_k!}$$

is given by

$$\mathrm{Map}_*(S^2 K_n, \Sigma^\infty((SX_1)^{\wedge n_1} \wedge \cdots \wedge (SX_k)^{\wedge n_k}))_{h(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k})},$$

where the action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ on K_n is given by restriction from the action of Σ_n and the action on $(SX_1)^{\wedge n_1} \wedge \cdots \wedge (SX_k)^{\wedge n_k}$ is the obvious one (this analogy with ordinary calculus is addressed in Lemma 1.3 and the explanation following it). The second way is to use the Hilton–Milnor theorem. According to this theorem, $\Omega S(X_1 \vee X_2 \vee \cdots \vee X_k)$ is homotopy-equivalent to a weak infinite product $\prod_i \Omega S Y_i$, where Y_i are smash products of some of the X_js. Evaluating the $(n_1, \dots, n_k)$th differential of $\prod_i \Omega S Y_i$ and comparing the two outcomes yields the following theorem.

Theorem 0.1. *For any connected based spaces $X_1, \dots, X_k$, there is an equivalence of spectra, which is natural in $X_1, \dots, X_k$:*

$$\begin{aligned} &\mathrm{Map}_*(S^2 K_n, \Sigma^\infty((SX_1)^{\wedge n_1} \wedge \cdots \wedge (SX_k)^{\wedge n_k}))_{h(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k})} \\ &\quad \simeq \bigvee_{d | n_0} \left(\bigvee_{B(n_1/d, \dots, n_k/d)} \mathrm{Map}_*(S^2 K_d, \Sigma^\infty(S(X_1)^{\wedge n_1/d} \wedge \cdots \wedge (X_k)^{\wedge n_k/d})^{\wedge d})_{h\Sigma_d} \right). \end{aligned}$$

(Here $n_0 = \gcd(n_1, \dots, n_k)$ and the definition of $B(n_1/d, \dots, n_k/d)$ is given in Eq. (3) *below.)*

Next we take each one of the spaces X_j to be a wedge of i_j copies of S^{2m}, an even-dimensional sphere (m is arbitrary, but it is the same for all j). Thus, for $j = 1, \dots, k$ we take $X_j = \bigvee_{j=1}^{i_j} S^{2m}$ where i_j is some positive integer. Passing to homology we obtain the following theorem as a corollary.

Theorem 0.2. *Let n_0 and $B(n_1/d,\dots,n_k/d)$ be as in the previous theorem. For any ground ring R, and any finitely generated free R-modules $N_1,\dots,N_k$*

$$H_*(\Sigma_{n_1}\times\cdots\times\Sigma_{n_k};\mathscr{L}ie_R(n)\otimes_R N_1^{\otimes n_1}\otimes_R\cdots\otimes_R N_k^{\otimes n_k})$$

$$\cong\bigoplus_{d|n_0}\left(\bigoplus_{B(n_1/d,\dots,n_k/d)} H_*(\Sigma_d;\mathscr{L}ie_R(d)\otimes_R(N_1^{\otimes n_1/d}\otimes_R\cdots\otimes_R N_k^{\otimes n_k/d})^{\otimes d})\right).$$

Now take the ground ring to be $\mathbb{Z}/p$ for some prime p, and let $N_1\cong\cdots\cong N_k\cong R$ (which corresponds to taking all the spaces X_j in Theorem 0.1 to be S^{2m}). The groups $H_*(\Sigma_d;\mathscr{L}ie(d))$ have been computed in [1]. In particular, if d is not a power of p, then $H_*(\Sigma_d;\mathscr{L}ie(d))\cong 0$. If $d=p^i$ for some non-negative integer i, then let us denote $M_i=H_*(\Sigma_{p^i};\mathscr{L}ie(p^i))$. We will describe M_i explicitly in Section 3. Recall that $n_0=\gcd(n_1,\dots,n_k)$ and let us write $n_0=mp^j$ where p does not divide m. Then the only values of d in Theorem 0.2 that we need to consider are those that are powers of p. Therefore,

$$H_*(\Sigma_{n_1}\times\cdots\times\Sigma_{n_k};\mathscr{L}ie(n))\cong\bigoplus_{i=0}^{j}M_i^{(p^i/n)\sum_{e|mp^{j-i}}\mu(e)\frac{(n/ep^i)!}{(n_1/ep^i)!\cdots(n_k/ep^i)!}}.$$

Here μ denotes the number-theoretic Möbius function. Because of the properties of μ, the only values of e we need to consider in the exponent of the right-hand side are m' and (if $j-i>0$) $m'p$ where $m'|m$. We obtain the following theorem

Theorem 0.3. *Let $n=n_1+\cdots+n_k$. Let $n_0=\gcd(n_1,\dots,n_k)$. Let the ground ring be $\mathbb{Z}/p$. Write $n_0=mp^j$ where p does not divide m. Then*

$$H_*(\Sigma_{n_1}\times\cdots\times\Sigma_{n_k};\mathscr{L}ie(n))$$

$$\cong\bigoplus_{i=0}^{j-1}M_i^{(p^i/n)\sum_{m'|m}\mu(m')\left(\frac{(n/m'p^i)!}{(n_1/m'p^i)!\cdots(n_k/m'p^i)!}-\frac{(n/m'p^{i+1})!}{(n_1/m'p^{i+1})!\cdots(n_k/m'p^{i+1})!}\right)}$$

$$\oplus M_j^{(p^j n)\sum_{m'|m}\mu(m')\frac{(n/m'p^j)!}{(n_1/m'p^j)!\cdots(n_k/m'p^j)!}}.$$

The paper is organized as follows: in Section 1 we present a very brief survey of some of Goodwillie's theory of Taylor towers and state it in the multi-variable form that we need. All the material in this section is either quoted from [7] or is adapted from there in a straightforward way. In Section 2 we recall various facts from algebraic topology that we need. In Section 3, we prove Theorems 0.1 and 0.2 and also describe the M_i's of Theorem 0.3.

1. Multivariable calculus

Let $\mathscr{U}$ denote the category of spaces with a non-degenerate basepoint. We will be concerned with calculus of homotopy functors $F:\mathscr{U}^{\times k}\to\mathscr{U}$. We begin with an overview of the case $k=1$, which is the subject of [5–7]. According to Goodwillie's calculus, certain functors are analogous to polynomial functions. Namely, a functor is analogous to a polynomial of degree n if it is n-excisive in the sense of [6, Definition 3.1]. One of the main theorems of [7] is that if a functor F is ρ-analytic in the sense of [6, Definition 4.2], then there exists a tower of functors $\{P_nF\}_{n\geq 0}$ with natural transformations $\cdots\to P_nF\to P_{n-1}F\to\cdots$ such that P_nF is of degree n and there exist natural transformations $p_n:F\to P_nF$ which are the unique best approximations of F by n-excisive functors in some precise sense. Since [7] has not been published yet, we spell out the details of the construction, but not of the proofs.

Let $C(n)$ denote the category whose objects are subsets of $\underline{n}=\{1,\dots,n\}$ and whose morphisms are inclusions. Let $C_0(n)$ denote the full subcategory of $C(n)$ whose objects are the non-empty subsets of $\underline{n}$. Let $\mathscr{C}$ be a category. An n-dimensional cubical diagram in $\mathscr{C}$ is a functor $C(n)\to\mathscr{C}$. A punctured n-dimensional cubical diagram is a functor $C_0(n)\to\mathscr{C}$. For our purposes, $\mathscr{C}$ will always be any well-behaved version of the category of either based spaces or spectra.

P_n involves the infinite iteration of another construction T_n. For $X\in\mathscr{U}$ the diagram $U\mapsto X*U$, $U\subseteq\underline{n+1}$ (here $*$ denotes join) determines a strongly co-cartesian $(\underline{n+1})$-cube [6, Definition 2.1] and a map, as in [6, Definition 1.2],

$$F(X)\to(T_n^1F)(X)\overset{\text{def.}}{=}\operatorname{holim}\{F(X*U)\mid U\in C_0(n+1)\}.$$

For $i>1$, define the functor T_n^iF inductively by $T_n^iF=T_n^1T_n^{i-1}F$. It is easy to see that

$$T_n^iF(X)\cong\operatorname{holim}\{F(X*U_1*\cdots*U_i)\mid U_1\times\cdots\times U_i\in C_0(n+1)^i\}$$

and we might as well have taken this to be the definition. We will do this when we define the multi-variable analogue of T_n^i. Clearly, there are natural transformations $T_n^iF\to T_n^{i+1}F$. P_nF is defined to be the homotopy colimit of the diagram

$$F=T_n^0F\to T_n^1F\to T_n^2F\to\cdots\to T_n^iF\to\cdots.$$

It is shown in [7] that P_nF is n-excisive and the map $p_n:F\to P_nF$ is characterized, up to weak homotopy equivalence, among natural transformations from F to n-excisive functors by the property that $p_n(X):F(X)\to P_nF(X)$ is $((k-\rho)(n+1)-c)$-connected, where k is the connectivity of X and c is a constant which does not depend on X or n.

According to [7], this tower should be thought of as the Taylor expansion of F at the one-point space (but we will subsequently call it the Goodwillie tower of F). Thus, P_nF is the nth "Taylor polynomial" of F and in particular P_0F is a (homotopy) constant functor with $P_0F(X)\simeq F(*)$ for all X. Note also that if F is reduced, i.e $F(*)\simeq *$, then $T_1F(X)\simeq\Omega F(SX)$ and P_1F, the linearization of F, is, essentially,

$\Omega^\infty F(S^\infty X)$. Let D_nF denote the homotopy fiber of the map $P_nF \to P_{n-1}F$. This is a well-defined functor and we call it the nth differential of F. It can be shown that D_nF is homogeneous of degree n in the sense that it is excisive of degree n and $P_{n-1}D_nF \simeq *$. Homogeneous functors were classified in [7]: a homogeneous functor of degree n is determined, at least up to homotopy, by a spectrum A endowed with an action of the symmetric group Σ_n and (up to weak homotopy equivalence) it has the form $\Omega^\infty((A \wedge X^{\wedge n})_{h\Sigma_n})$. Note the visual resemblance to the formula $ax^n/n!$ for the nth summand in a Taylor expansion. If $D_nF(X) \simeq \Omega^\infty((A \wedge X^{\wedge n})_{h\Sigma_n})$, then we say that the spectrum A, together with the action of Σ_n, is the nth derivative of F. Note also that for $n=1$ one gets the classical description of a generalized homology theory as given by the Brown representability theorem.

For a general k, we will say that a functor $F: \mathcal{U}^{\times k} \to \mathcal{U}$ is analytic if it is analytic considered in each variable separately. For a multi-index $\boldsymbol{n} = (n_1, \ldots, n_k)$, we say that F is $\boldsymbol{n}$-excisive if it is n_i-excisive considered as a functor of X_i for any choice of fixed $X_1, \ldots, \hat{X}_i, \ldots, X_k$. We say that F is multilinear if it is reduced and linear (1-excisive) in each variable separately. Given a subgroup H of Σ_k we say that F is symmetric with respect to H if it extends to a functor $F: H \wr \mathcal{U}^{\times k} \to \mathcal{U}$. If $H = \Sigma_k$, then we simply say that F is symmetric. It is shown in [7] that homogeneous one-variable functors of degree k are equivalent to symmetric multilinear functors of k variables. In fact, up to weak homotopy equivalence, a symmetric multilinear functor has the form $\Omega^\infty(A \wedge X_1 \wedge \cdots \wedge X_k)$, where A is a spectrum with an action of Σ_k.

By a k-index, or simply a multi-index, when k is clear from the context, we mean an ordered k-tuple of non-negative integers. We denote multi-indices with bold letters such as $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{n}$. Given multi-indices $\boldsymbol{i} = (i_1, \ldots, i_k)$ and $\boldsymbol{j} = (j_1, \ldots, j_k)$, we say that $\boldsymbol{i} \le \boldsymbol{j}$ if $i_l \le j_l$ for $l = 1, \ldots, k$. For a functor $F: \mathcal{U}^{\times k} \to \mathcal{U}$, and multi-indices $\boldsymbol{i} = (i_1, \ldots, i_k)$ and $\boldsymbol{n} = (n_1, \ldots, n_k)$ we define

$$\begin{aligned} T^{\boldsymbol{i}}_{\boldsymbol{n}} F(X_1, \ldots, X_k) = \ & \mathrm{holim}\{F(X_1 * U^1_1 * U^1_2 * \cdots * U^1_{i_1}, \ldots, X_k * U^k_1 * \cdots * U^k_{i_k}) \mid \\ & (U^1_1, \ldots, U^1_{i_1}, \ldots, U^k_{i_k}) \in C_0(n_1+1)^{i_1} \times \cdots \times C_0(n_k+1)^{i_k}\}. \end{aligned}$$

By analogy with the one-variable case, it is clear that if $\boldsymbol{i}_1 \le \boldsymbol{i}_2$ then there is a canonical natural transformation $T^{\boldsymbol{i}_1}_{\boldsymbol{n}} F \to T^{\boldsymbol{i}_2}_{\boldsymbol{n}} F$ and if also $\boldsymbol{i}_2 \le \boldsymbol{i}_3$, then the map $T^{\boldsymbol{i}_1}_{\boldsymbol{n}} F \to T^{\boldsymbol{i}_3}_{\boldsymbol{n}} F$ is the composition $T^{\boldsymbol{i}_1}_{\boldsymbol{n}} F \to T^{\boldsymbol{i}_2}_{\boldsymbol{n}} F \to T^{\boldsymbol{i}_3}_{\boldsymbol{n}} F$. We define $P_{\boldsymbol{n}}$ to be the homotopy direct limit of the thus obtained k-dimensional diagram of all $T^{\boldsymbol{i}}_{\boldsymbol{n}}$. Intuitively,

$$P_{(n_1, \ldots, n_k)} F = T^{(\infty, \ldots, \infty)}_{(n_1, \ldots, n_k)} F.$$

Clearly, $P_{\boldsymbol{n}}F$ is $\boldsymbol{n}$-excisive. Let $\mathbf{1}_j$ denote the multi-index $(0, \ldots, 1, \ldots, 0)$ having a 1 at the j-th place and zeros elsewhere. Just as in the 1-variable case there is a map $P_n \to P_{n-1}$ induced by restriction of homotopy inverse limits, in the k-variables case there are maps $P_{\boldsymbol{n}} \to P_{\boldsymbol{n}-\mathbf{1}_j}$. Thus, the functors $P_{\boldsymbol{n}}F$ fit into a "k-dimensional" inverse limit system, that converges to F in an appropriate sense. When we speak of $P_{\boldsymbol{n}}F$, we allow that some of the indices of $\boldsymbol{n}$ are ∞, in which case we mean that we take the inverse limit with respect to these indices. For example, $P_{(\infty,n_2,\dots,n_k)} = \operatorname{holim}_{n_1\to\infty} P_{(n_1,n_2,\dots,n_k)}$.

In the case of a one-variable functor F, there is an important relation between the nth differential of F and the nth cross-effect of F. The connection is explained in [7]. We present a brief summary. Given a functor $F:\mathscr{U}\to\mathscr{U}$, the nth cross-effect of F, denoted $\chi_n F$, is a functor of n variables defined as follows: For based spaces $X_1,\dots,X_n$, we define an n-dimensional cubical diagram $\mathscr{S}(X_1,\dots,X_n)$ by

$$\underline{n}\backslash T \mapsto \bigvee_{s\in T} X_s$$

with the maps in the cube being the obvious retractions. Then $\chi_n F(X_1,\dots,X_n)$ is defined to be the iterated homotopy fiber (or the total fiber, as defined in [7, Definitions 1.1–1.2a]) of $\mathscr{S}(X_1,\dots,X_n)$. It is easily seen from the definitions that $\chi_n F$ is symmetric and reduced. It follows that $P_{(1,1,\dots,1)}\chi_n F(X_1,\dots,X_n)$, the multilinearization of $\chi_n F$, is given, up to homotopy, by

$$\operatorname*{hocolim}_{k_1,\dots,k_n\to\infty} \Omega^{k_1+\cdots+k_n}\chi_n F(S^{k_1}X_1,\dots,S^{k_n}X_n).$$

We denote this functor $D^{(n)}F$. $D^{(n)}F$ is a symmetric multilinear functor, and as such is represented by a spectrum with an action of Σ_n; denote this spectrum by $\boldsymbol{A}_n$. The following result is proved in [7]

Lemma 1.1. *If $\boldsymbol{A}_n$ is as above, then $\boldsymbol{A}_n$ is the nth derivative of F.*

Let $F:\mathscr{U}^k\to\mathscr{U}$ be a functor of k variables. Let $\boldsymbol{n}=(n_1,\dots,n_k)$. We define $D_{\boldsymbol{n}}F$ to be the iterated homotopy fiber of the k-dimensional cube given by $U\mapsto P_{\boldsymbol{n}-\mathbf{1}_U}$, where $\mathbf{1}_U$ is defined to be the multi-index which has in its jth place 1 or 0 depending on whether $j\in U$ or $j\notin U$ respectively. $D_{\boldsymbol{n}}F$ is $\boldsymbol{n}$-homogeneous in the appropriate sense. There is a natural (weak) homotopy equivalence

$$D_{\boldsymbol{n}}F(X_1,\dots,X_k)\simeq \Omega^\infty(\boldsymbol{A}\wedge X_1^{\wedge n_1}\wedge\cdots\wedge X_k^{\wedge n_k})_{h(\Sigma_{n_1}\times\Sigma_{n_2}\times\cdots\times\Sigma_{n_k})},$$

where $\boldsymbol{A}$ is a spectrum with an action of $\Sigma_{n_1}\times\Sigma_{n_2}\times\cdots\times\Sigma_{n_k}$. We may think of $\boldsymbol{A}$, together with the action, as the $\boldsymbol{n}$th derivative of F. Again, we would like to establish the relation between derivatives and cross-effects. So, given F as before, we may form its n_1th cross-effect with respect to the variable X_1, n_2th cross-effect in the second variable, and so on. One can easily see that the outcome does not depend on the order of variables. The outcome is a functor of $n_1+n_2+\cdots+n_k$ variables, which is reduced and symmetric with respect to an action of $\Sigma_{n_1}\times\Sigma_{n_2}\times\cdots\times\Sigma_{n_k}$. Upon multilinearizing, we obtain a functor which is multilinear and is symmetric with respect to $\Sigma_{n_1}\times\Sigma_{n_2}\times\cdots\times\Sigma_{n_k}$. As such, it is represented by a spectrum with an action of $\Sigma_{n_1}\times\Sigma_{n_2}\times\cdots\times\Sigma_{n_k}$. Denote it by $\boldsymbol{A}_{\boldsymbol{n}}$.

Lemma 1.2. *If $\boldsymbol{A}_{\boldsymbol{n}}$ is as above, then $\boldsymbol{A}_{\boldsymbol{n}}$ is the $(n_1,\dots,n_k)$th derivative of F.*

Proof. Very similar to the proof of our Lemma 1.1 as given in [7]. □

Lemma 1.3. *Let $F(X_1,\dots,X_k)=G(\bigvee_{i=1}^k X_i)$, where G is some functor of one variable. Let $\boldsymbol{n}=(n_1,\dots,n_k)$ and let $n=n_1+\cdots+n_k$. Then the* **n***th derivative of F is equivalent to the nth derivative of G with the action of $\Sigma_{n_1}\times\Sigma_{n_2}\times\cdots\times\Sigma_{n_k}$ obtained by restriction from the action of Σ_n.*

Explanation. Let $f(x_1,\dots,x_k)$ be an analytic function, say, from $\mathbb{R}^k$ to $\mathbb{R}$. Assume, furthermore, that $f(x_1,\dots,x_k)=g(x_1+\cdots+x_k)$ for some $g:\mathbb{R}\to\mathbb{R}$. Then

$$\frac{\partial^n f}{\partial x_1^{n_1}\cdots\partial x_k^{n_k}}(0,\dots,0)\frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1!\cdots n_k!}=\frac{\mathrm{d}^n g}{\mathrm{d}x^n}(0)\frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1!\cdots n_k!}.$$

Proof of Lemma 1.3. Immediate from Lemmas 1.1 and 1.2. □

Lemma 1.4. *Let $F(X_1,\dots,X_k)=G(X_1^{\wedge m_1}\wedge X_2^{\wedge m_2}\wedge\cdots\wedge X_k^{\wedge m_k})$. Then $D_{\boldsymbol{n}}F$ is trivial if n_i/m_i is not the same for all $i=1$. Otherwise,*

$$D_{\boldsymbol{n}}F(X_1,\dots,X_k)\simeq D_l G(X_1^{\wedge m_1}\wedge X_2^{\wedge m_2}\wedge\cdots\wedge X_k^{\wedge m_k}),$$

where $l=n_i/m_i$.

Explanation. Let $f(x_1,\dots,x_k)$ be an analytic function. Assume, furthermore, that $f(x_1,\dots,x_k)=g(x_1^{m_1}\cdot x_2^{m_2}\cdots x_k^{m_k})$ for some g. Then the Taylor series for g evaluated at $x_1^{m_1}\cdot x_2^{m_2}\cdots x_k^{m_k}$ is clearly the multi-variable Taylor expansion of f and it can be easily concluded that if n_i/m_i is not a constant non-negative integer independent of i, then

$$\frac{\partial^n f}{\partial x_1^{n_1}\cdots\partial x_k^{n_k}}(0,\dots,0)\frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1!\cdots n_k!}=0$$

and if $n_i/m_i=l$ for all i, then

$$\frac{\partial^n f}{\partial x_1^{n_1}\cdots\partial x_k^{n_k}}(0,\dots,0)\frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1!\cdots n_k!}=\frac{\mathrm{d}^l g}{\mathrm{d}x^l}(0)\frac{(x_1^{m_1}\cdots x_k^{m_k})^l}{l!}.$$

Proof of Lemma 1.4. In fact, we want to show that for any multi-index $\boldsymbol{n}$,

$$P_{\boldsymbol{n}}F(X_1,\dots,X_k)\simeq P_{(lm_1,\dots,lm_k)}F(X_1,\dots,X_k)\simeq P_l G(X_1^{\wedge m_1}\wedge X_2^{\wedge m_2}\wedge\cdots\wedge X_k^{\wedge m_k}),$$

where $l=\min\{\lfloor n_i/m_i\rfloor\}$. Without loss of generality, we may assume that $l=\lfloor n_1/m_1\rfloor$. Consider the functor $P_{(n_1,\infty,\dots,\infty)}F$, which amounts to considering F as a functor of X_1 only, and taking the n_1th Taylor polynomial in this variable. $P_{(n_1,\infty,\dots,\infty)}F$ is n_1-excisive (in X_1) and the map $F(X_1,\dots,X_k)\to P_{(n_1,\infty,\dots,\infty)}F(X_1,\dots,X_k)$ is $((n_1+1)(d_1-\rho)-c)$-connected, where d_1 is the connectivity of X_1 and ρ and c are some numbers, which depend on $X_2,\dots,X_k$, but not on X_1. On the other hand, consider the functor

$P_lG(X_1^{\wedge m_1} \wedge X_2^{\wedge m_2} \wedge \cdots \wedge X_k^{\wedge m_k})$ as a functor of X_1 only. It is obvious that it is lm_1-excisive and therefore n_1-excisive, since $n_1 \geq lm_1$. The natural map

$$F(X_1,\ldots,X_k) = G(X_1^{\wedge m_1} \wedge \cdots \wedge X_k^{\wedge m_k}) \to P_lG(X_1^{\wedge m_1} \wedge \cdots \wedge X_k^{\wedge m_k})$$

is

$$\begin{aligned}(l+1)(m_1d_1 + \cdots + m_kd_k - \rho') - c' &\geq ((l+1)m_1)(d_1 - \rho') - c' \\ &\geq (n_1+1)(d_1 - \rho') - c'\end{aligned}$$

connected, for some constants ρ' and c'. By universality properties of the Taylor approximations it follows that there is a weak homotopy equivalence

$$P_{(n_1,\infty,\ldots,\infty)}F(X_1,\ldots,X_k) \simeq P_lG(X_1^{\wedge m_1} \wedge X_2^{\wedge m_2} \wedge \cdots \wedge X_k^{\wedge m_k}). \tag{1}$$

Now consider $P_{(n_1,\ldots,n_k)}F$. It is weakly equivalent to $P_{(\infty,n_2,\ldots,n_k)}P_{(n_1,\infty,\ldots,\infty)}F$. But by (1), $P_{(n_1,\infty,\ldots,\infty)}F$ is lm_i-excisive in the variable X_i for $i = 1,2,\ldots,k$. Since we defined l to be $\min\{\lfloor n_i/m_i \rfloor\}$, it follows that $n_i \geq lm_i$, and therefore $P_{(n_1,\infty,\ldots,\infty)}F$ is n_i-excisive in the variable X_i for $i = 2,\ldots,k$. Hence,

$$P_{(n_1,\ldots,n_k)}F(X_1,\ldots,X_k) \simeq P_lG(X_1^{\wedge m_1} \wedge X_2^{\wedge m_2} \wedge \cdots \wedge X_k^{\wedge m_k}).$$

The lemma readily follows. □

2. Miscellaneous preliminary results

Let W be a based space or a spectrum with an action of Σ_n. Let R be a commutative ring. Let $\widetilde{H}_*(-;R)$ denote reduced homology with coefficients in R.

Fact 2.1. *Suppose that $\widetilde{H}_*(W;R)$ is concentrated in degree i. Then*

$$\widetilde{H}_*(W_{h\Sigma_n};R) \cong H_{*+i}(\Sigma_n;\widetilde{H}_i(W;R)).$$

We will use the following "geometric realization" of $\mathscr{L}ie(n)$. Let k_n be the category of unordered partitions of $\underline{n}$. Thus, the objects of k_n are unordered partitions of $\underline{n}$, and there is a morphism $\lambda_1 \to \lambda_2$ iff λ_2 is a refinement of λ_1. The category of partitions that we define here is the opposite of the category of partitions as defined in [1], but it makes no difference since we only are interested in the simplicial nerve of k_n. Obviously, k_n has an initial and a final object. Denote these $\hat{0}$ and $\hat{1}$, respectively. Let $\tilde{k}_n = k_n \backslash \{\hat{0},\hat{1}\}$. Let $\widetilde{K}_n$ be the geometric realization of the simplicial nerve of $\tilde{k}_n$. Let K_n be the unreduced suspension of $\widetilde{K}_n$. Obviously, K_n has an action of Σ_n. Let S^n denote the n-dimensional sphere.

Fact 2.2. *Non-equivariantly*

$$K_n \simeq \bigvee_{(n-1)!} S^{n-2}.$$

In particular, the reduced homology of K_n is concentrated in degree $n-2$.

Fact 2.3. *As a Σ_n-representation, $\mathscr{L}ie(n) \cong \hom(\widetilde{H}_{n-2}(K_n), Z[-1])$.*

Proof. It is known [2, Theorem 12.3, p. 302] that $\mathscr{L}ie(n)$ is isomorphic, as a Σ_n-representation, to the top homology of $F(R^{2k+1};n)$, the space of ordered configurations of n points in an odd-dimensional Euclidean space (k is arbitrary). Now view $F(R^{2k+1};n)$ as a Σ_n-equivariant subspace of $R^{(2k+1)n}$ and also, by adding a point at infinity, of $S^{(2k+1)n}$. The top homology group of $S^{(2k+1)n}$ is isomorphic to $Z[-1]$ as a Σ_n-representation. Let $\Delta^n S^{2k+1}$ denote the complement of $F(R^{2k+1};n)$ in $S^{(2k+1)n}$. Let H_{b} denote the bottom reduced homology of $\Delta^n S^{2k+1}$. It follows by Alexander duality that H_{b} is in dimension $n+2k-1$ and that $\mathscr{L}ie(n) \cong_{Z[\Sigma_n]} \hom(H_{\mathrm{b}}, Z[-1])$. It remains to show that $H_{\mathrm{b}} \cong \widetilde{H}_{n-2}(K_n)$ as a Σ_n-representation. It is enough to prove the following proposition (the first author is grateful to J. Rognes for explaining it to him).

Proposition 2.4. *There is a Σ_n-equivariant (weak) map*

$$\Delta^n S^l \to S^l K_n$$

which, for large enough l, induces an isomorphism on the bottom homology of $\Delta^n S^l$.

The rest of the proof of the fact is occupied by the proof of the proposition. Consider the covering of $\Delta^n S^l$ by the subspaces $U_{i,j}$, $1 \le i < j \le n$, where

$$U_{i,j} = \{(s_1 \wedge \cdots \wedge s_n) \in \Delta^n S^l \mid s_i = s_j\}.$$

The point is that the intersection poset of this covering is isomorphic to the poset of unordered partitions of $\underline{n}$ and the intersections in this covering are highly enough connected. In more detail, let $A = \{(i_1,j_1),(i_2,j_2),\ldots,(i_M,j_M)\}$ be a collection of pairs $1 \le i_m < j_m \le n$. Let $U_A = \bigcap_{(i_m,j_m)\in A} U_{i_m,j_m}$. We associate with A a graph on n vertices, labeled $1,\ldots,n$, as follows: There is an edge (i,j) iff $(i,j)\in A$. The connected components of this graph determine a partition of $\underline{n}$. Clearly, U_A depends only on the partition associated with A. Thus, the poset associated with the covering of $\Delta^n S^l$ by $U_{i,j}$ is isomorphic to $k_n \backslash \hat{1}$. Let J be the intersection diagram of this covering. Thus, J is a functor

$$k_n \backslash \hat{1} \to \text{Spaces}$$

given by

$$\lambda \to \{(s_1 \wedge \cdots \wedge s_n) \in \Delta^n S^l \mid s_i = s_j \text{ if } i \text{ and } j \text{ are in the same component of } \lambda\}.$$

Thus, $\Delta^n S^l$ is the strict direct limit of J. Let $\Delta^n_h S^l$ be the *homotopy* direct limit of J. There is a canonical Σ_n-equivariant map

$$\Delta^n_h S^l \to \Delta^n S^l$$

which is a homotopy equivalence.

We now define another functor $\tilde{J}: k_n \backslash \{\hat{1}\} \to$ Spaces as follows:

$$\lambda \to \left\{ (s_1 \wedge \cdots \wedge s_n) \in \Delta^n S^l \,\middle|\, s_i = s_j \text{ if } i \text{ and } j \text{ are in the same component of } \lambda, \right.$$
$$\left. \sum_{i=1}^{n} s_i = 0 \text{ and } \sum_{i=1}^{n} \|s_i\|^2 = 1 \right\}.$$

Here we think of $\Delta^n S^l$ as a subspace of S^{nl}, which is the one-point compactification of $\mathbb{R}^{nl}$. The functor $\tilde{J}$ takes values in subspaces of $\mathbb{R}^{nl}$. Notice that $\tilde{J}(\hat{0}) = \emptyset$ and therefore $\operatorname{hocolim} \tilde{J} \cong \operatorname{hocolim} \tilde{J}|_{\tilde{k}_n}$. In general, if λ has k components, then $\tilde{J}(\lambda) \cong S^{lk-l-1}$. Let $\tilde{\Delta}^n_h S^l$ denote the homotopy direct limit of $\tilde{J}$. It is easy to see that

$$\Delta^n_h S^l \cong S^l * \tilde{\Delta}^n_h S^l,$$

where $*$ denotes join. Next we note that since $\tilde{\Delta}^n_h S^l$ is the homotopy direct limit of a functor $\widetilde{k}_n \to$ Spaces and $\widetilde{K}_n$ is homeomorphic to the homotopy direct limit of the constant functor $\widetilde{k}_n \to *$, there is a map

$$\tilde{\Delta}^n_h S^l \to \widetilde{K}_n$$

which is $\sim l$-connected, since the values of $\tilde{J}$ are all $\sim l$-connected. All of the above assembles into the following chain of equivariant maps:

$$\Delta^n S^l \xleftarrow{\simeq} \Delta^n_h S^l \xrightarrow{\cong} S^l * \tilde{\Delta}^n_h S^l \to S^l * \widetilde{K}_n \xrightarrow{\simeq} S^l \wedge K_n.$$

The composed map is $\sim 2l$-connected since the map $S^l * \tilde{\Delta}^n_h S^l \to S^l * \widetilde{K}_n$ is. It follows that the map induces an isomorphism on the bottom homology for l large enough. This completes the proof. □

The space K_n plays an important role in calculus of functors. Consider the identity functor from based spaces to based spaces. The following is proved in [8] in a slightly different form:

Fact 2.5.

$$D_n \operatorname{Id}(X) \simeq \Omega^\infty(\operatorname{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}).$$

Proof. As mentioned above, this is a reformulation of the main theorem of [8]. For proof of the equivalence with the result in [8], see [1, Section 2]. □

Corollary 2.6.

$$D_n(\Omega S)(X) \simeq \Omega^\infty(\operatorname{Map}_*(S^2 K_n, \Sigma^\infty (SX)^{\wedge n})_{h\Sigma_n}).$$

Proof. For any functor $F: \mathscr{U} \to \mathscr{U}$, consider the functor $F': X \mapsto \Omega F(SX)$. It follows immediately from the universal properties of P_n that $(P_n F')(X) \simeq \Omega(P_n F)(SX)$ and

$(D_nF')(X) \simeq \Omega(D_nF)(SX)$. The corollary follows by letting F to be the identity functor. $\square$

Remark 2.7. It may seem that there is a problem with the definition of K_1. In fact, there is not. Either by careful checking of the definitions, or by convention, define K_1 to be the empty set. Then SK_1 should be interpreted as S^0, provided the suspension is unreduced. With these conventions, Fact 2.5 and Corollary 2.6 hold for $n = 1$.

3. Proof of the main theorems

Let $X_1, \ldots, X_k$ be sufficiently nice topological spaces, for example, connected CW-complexes with non-degenerate basepoints. By the Hilton–Milnor theorem [10] there is a natural homotopy equivalence

$$h\colon \prod_{i \in I} \Omega S Y_i \to \Omega S \bigvee_{j=1}^{k} X_j, \tag{2}$$

where I denotes the set of basic products on k letters. Essentially, each Y_i is a smash product of some of the spaces $X_1, \ldots, X_k$ corresponding to the basic product i. By the formula due to Witt [11] the number of basic products involving X_j exactly q_j times is

$$B(q_1, \ldots, q_k) = \frac{1}{q} \sum_{d | q_0} \mu(d) \frac{(q/d)!}{(q_1/d)! \cdots (q_k/d)!}, \tag{3}$$

where q_0 is the greatest common divisor of the numbers $q_1, \ldots, q_k$, $q = q_1 + \cdots + q_k$ and μ is the number theoretic Möbius function.

Let us denote $F_i(X_1, \ldots, X_k) = \Omega S Y_i$ and $G(X_1, \ldots, X_k) = \Omega S \bigvee_{j=1}^{k} X_j$. Let F be $\prod_{i \in I} F_i$. Let $\boldsymbol{n} = (n_1, \ldots, n_k)$, where $n_1, \ldots, n_k$ are positive integers and let $n_1 + \cdots + n_k = n$. The natural transformation h of (2) induces an equivalence

$$D_{\boldsymbol{n}} h : D_{\boldsymbol{n}} F \to D_n G.$$

Let I' denote the set of basic products which involve X_j at most n_j times for $j = 1, \ldots, k$. Clearly, I' is finite. Let $I'' = I \backslash I'$. Then $F = \prod_{i' \in I'} F_{i'} \times \prod_{i'' \in I''} F_{i''}$.

The operator $D_{\boldsymbol{n}}$ commutes, up to homotopy, with finite products of functors. Therefore,

$$D_{\boldsymbol{n}} F \simeq D_{\boldsymbol{n}} \prod_{i' \in I'} F_{i'} \times D_{\boldsymbol{n}} \prod_{i'' \in I''} F_{i''}.$$

Proposition 3.1.

$$D_{\boldsymbol{n}} \prod_{i'' \in I''} F_{i''} \simeq *.$$

Proof. It follows immediately from the definition of I'' and Lemma 1.4 that for any $i'' \in I''$, $D_n F_{i''} \simeq *$. The proposition does not follow automatically because in general D_n commutes only with finite inverse homotopy limits. However, D_n commutes with arbitrary filtered homotopy direct limits, and since the product is a weak product, it can be written as a homotopy direct limit of finite products. □

On the other hand, I' is finite, and therefore there is a weak equivalence

$$D_{(n_1,\dots,n_k)}F \simeq \prod_{i \in I'} D_{(n_1,\dots,n_k)}F_i \simeq D_{(n_1,\dots,n_k)}G. \tag{4}$$

Evaluating the right-hand side of (4) at $(X_1,\dots,X_k)$ and using Lemma 1.3, we obtain

$$\begin{aligned} D_{(n_1,\dots,n_k)}G(X_1,\dots,X_k) &= D_{(n_1,\dots,n_k)}\left(\Omega S \bigvee_{j=1}^{k} X_j\right) \\ &= \Omega^\infty \mathrm{Map}_*(S^2K_n, \Sigma^\infty((SX_1)^{\wedge n_1} \wedge \cdots \wedge (SX_k)^{\wedge n_k}))_{h(\Sigma_{n_1}\times\cdots\times\Sigma_{n_k})}. \end{aligned} \tag{5}$$

Similarly, evaluating the middle part of (4) at $(X_1,\dots,X_k)$ and using Lemma 1.4, we obtain

$$\begin{aligned} \prod_{i \in I'} D_{(n_1,\dots,n_k)}F_i(X_1,\dots,X_k) &= \prod_{i \in I'} D_{(n_1,\dots,n_k)}\Omega S Y_i \\ &= \prod_{d|n_0} (\Omega^\infty \mathrm{Map}_*(S^2K_d, \Sigma^\infty(S(X_1)^{\wedge n_1/d} \wedge \cdots \wedge (X_k)^{\wedge n_k/d})^{\wedge d})_{h\Sigma_d})^{B(n_1/d,\dots,n_k/d)}, \end{aligned} \tag{6}$$

where n_0 is the greatest common divisor of the numbers $n_1,\dots,n_k$. Comparing (5) and (6), and passing from infinite loop spaces to spectra, we obtain Theorem 0.1.

Let us now take X_j to be a wedge of copies of an even-dimensional sphere S^{2m} for every j. Then, using Theorem 0.1, taking homology groups, and applying Facts 2.1 and 2.3, we obtain Theorem 0.2.

From here on, we assume that the ground ring is $\mathbb{Z}/p$. It remains to describe the M_i's of Theorem 0.3. We have indicated in the introduction that the M_i's were computed in [1]. To be more precise, the homology groups of the spectrum

$$\mathrm{Map}_*(S^2K_d, \Sigma^\infty(S^{(2mn/d)+1})^{\wedge d})_{h\Sigma_d}$$

with $\mathbb{Z}/p$ coefficients have been computed in [1]. In particular, if d is not a power of p, then

$$H_*(\mathrm{Map}_*(S^2K_d, \Sigma^\infty(S^{(2mn/d)+1})^{\wedge d})_{h\Sigma_d}; \mathbb{Z}/p) \cong 0.$$

Assume therefore $d = p^i$ for some non-negative integer i. The homology groups of $\mathrm{Map}_*(S^2K_d, \Sigma^\infty(S^{(2mn/d)+1})^{\wedge d})_{h\Sigma_d}$ are independent of m, up to a dimension shift, and we may as well take $m = 0$. Using Facts 2.1 and 2.3 once more, it is easy to see that

$$M_i \cong H_*(\mathrm{Map}_*(S^2K_d, \Sigma^\infty S^d)_{h\Sigma_d}; \mathbb{Z}/p).$$

Roughly speaking, up to a suitable suspension, M_i has a basis consisting of the "completely inadmissible" Dyer–Lashof words of length i (which is almost the same as the set of admissible Steenrod words of length i).

Theorem 3.2. *If $d = p^i$, then the following constitutes a basis for $\Sigma^{1+i}M_i$: if $p > 2$*

$$\{\beta^{\varepsilon_1}Q^{s_1}\cdots\beta^{\varepsilon_i}Q^{s_i}u \mid s^i \geq 1,\ s^j > ps^{j+1} - \varepsilon_{j+1}\ \forall 1 \leq j < i\},$$

if $p = 2$

$$\{Q^{s_1}\cdots Q^{s_i}u \mid s^i \geq 1,\ s^j > 2s^{j+1}\ \forall 1 \leq j < i\}.$$

Here u is of dimension 1, Q^{s_j}s are the Dyer–Lashof operations and βs are the homology Bocksteins [2]. *Thus Q^s increases dimension by s if $p = 2$ and by $2s(p-1)$ if $p > 2$ and β decreases dimension by one.*

References

[1] G.Z. Arone, M. Mahowald, The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres, preprint, available on the Hopf archive.
[2] F.R. Cohen, T.J. Lada, J.P. May, The homology of iterated loop spaces, Lecture Notes in Math., Vol. 533, Springer, Berlin, 1976.
[3] F.R. Cohen, L.R. Taylor, On the representation theory associated to the cohomology of configuration spaces, Contemp. Math. 146 (1993) 91–109.
[4] F.R. Cohen, L.R. Taylor, Computations of Gelfand–Fuks cohomology, the cohomology of function spaces and the cohomology of configuration spaces, in: Geometric Applications of Homotopy Theory I, Lecture Notes in Math., Vol. 657, Springer, Berlin, 1978, pp. 106–143.
[5] T.G. Goodwillie, Calculus I: the first derivative of pseudoisotopy theory, K-Theory 4 (1990) 1–27.
[6] T.G. Goodwillie, Calculus II: analytic functors, K-Theory 5 (1992) 295–332.
[7] T.G. Goodwillie, Calculus III: the Taylor series of a homotopy functor, preprint.
[8] B. Johnson, The derivatives of homotopy theory, Trans. Amer. Math. Soc. 347(4) (1995) 1295–1321.
[9] R.P. Stanley, Some aspects of groups acting on finite posets, J. Combin. Theory Ser. A 32 (1982) 132–161.
[10] G.W. Whitehead, Elements of Homotopy Theory, Springer, New York, 1978.
[11] E. Witt, Treue Darstellung Liescher Ringe, J. Reine Angew. Math. 177 (1937) 152–160.